Fall 2015 Math 245 Exam 2 Solutions

Problem 1. Carefully define each of the following terms:

a. gcd

The **gcd** or **greatest common divisor** of integers a, b, not both zero, is the largest integer that divides each of them.

b. (set) union

The **union** of two sets A, B is the set that consists of those elements in A, B, or both.

c. maximal

A poset element a is **maximal** if there isn't some different poset element b with $a \leq b$.

d. codomain

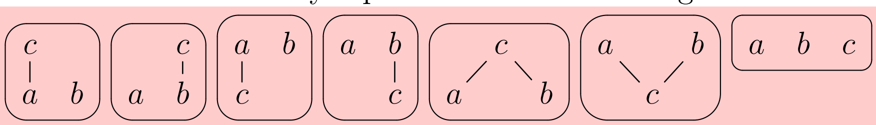
The **codomain** of a function is the set in which it takes its values. Alternatively, it is the second set of the direct product, from which the function relation is drawn.

e. bijection

A function is a **bijection** if it is both one-to-one and onto.

Problem 2. Consider the posets on $A = \{a, b, c\}$ where a, b are not comparable. Draw a Hasse diagram of each. Be sure to clearly separate the different diagrams.

There are seven:



Problem 3. For all sets A, B, C, prove that $(A \cap B) \setminus C \subseteq A \cup B$.

Let $x \in (A \cap B) \setminus C$. Then $x \in (A \cap B)$ and $x \notin C$. By conjunctive simplification we conclude that $x \in (A \cap B)$. Hence $x \in A$ and $x \in B$. Hence in particular $x \in A$ or $x \in B$. Since x was arbitrary, the desired result follows.

Problem 4. Let $A = \{a, b, c\}, B = \{a, b, d, e\}$. Prove or disprove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

The statement is false. To disprove, we need an element of $\mathcal{P}(A)$ that is not an element of $\mathcal{P}(B)$. That is, we need a specific subset of A that is not a subset of B. One natural choice is $x = \{c\}$. We have $x \in \mathcal{P}(A)$ but $x \notin \mathcal{P}(B)$. Hence $\mathcal{P}(A) \nsubseteq \mathcal{P}(B)$.

Problem 5. Let $A = \{a, b, c\}$. Give a relation on A that is simultaneously an equivalence relation *and* a partial order *and* a function.

There is only one such relation: $R = \{(a, a), (b, b), (c, c)\}$.

Problem 6. Use the (extended) Euclidean algorithm to first find $\gcd(33, 9)$, and then to express $\gcd(33, 9)$ as a linear combination of 33 and 9.

Step 1: $33 = 3 \cdot 9 + 6$. Step 2: $9 = 1 \cdot 6 + 3$. Step 3: $6 = 2 \cdot 3 + 0$. Hence gcd $= 3$, and we back-substitute. Step 4: $3 = 9 - 1 \cdot 6$. Step 5: $3 = 9 - 1 \cdot (33 - 3 \cdot 9) = 4 \cdot 9 - 1 \cdot 33$.

Problem 7. Let $A = \{a, b, c\}$. Find all partitions of A.

There are five: $\{a\} \cup \{b\} \cup \{c\}$, $\{a\} \cup \{b, c\}$, $\{b\} \cup \{a, c\}$, $\{c\} \cup \{a, b\}$, and $\{a, b, c\}$.

Problem 8. Prove or disprove: for all functions $f : \mathbb{R} \to \mathbb{R}$, if f is injective then f is surjective.

The statement is false. To disprove, we need a counterexample, a function that is injective but *not* surjective. Many solutions are possible; one is $f(x) = e^x$. It is not surjective because $f(x) > 0$ for all real x, so $-1 \in \mathbb{R}$ is not in the image of f. Lastly, we prove it is injective: if $f(a) = f(b)$ then $e^a = e^b$; taking logarithms we conclude that $a = b$.

Problem 9. Let S be a Boolean algebra. Prove that, for any $x \in S$, that $x \oplus 1 = 1$.

We have $x \oplus 1 = x \oplus (x \oplus \overline{x}) = (x \oplus x) \oplus \overline{x} = x \oplus \overline{x} = 1$. The first and last equalities are justified by a property of inverses in Boolean algebras, the second equality is justified by associativity of $\oplus$, and the third inequality is justified by the idempotency of $\oplus$.

Problem 10. Solve the recurrence $a_n = a_{n-1} + 6a_{n-2}$ with initial conditions $a_0 = 0, a_1 = 5$.

This has characteristic equation $t^2 = t + 6$, which factors as $(t - 3)(t + 2) = 0$. Hence the general solution is $a_n = A(3)^n + B(-2)^n$. The initial conditions give us $0 = a_0 = A3^0 + B(-2)^0 = A + B$ and $5 = a_1 = A3^1 + B(-2)^1 = 3A - 2B$. This has solution $\{A = 1, B = -1\}$, so our recurrence has solution $a_n = 3^n - (-2)^n$.